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# Elementary and Middle School Mathematics

*Teaching Developmentally*

NINTH EDITION



John A. Van de Walle • Karen S. Karp • Jennifer M. Bay-Williams

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G L O B A L   E D I T I O N

# Elementary and Middle School Mathematics

## Teaching Developmentally

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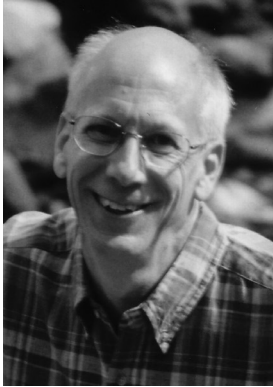
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## SECTION I Teaching Mathematics: Foundations and Perspectives

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The fundamental core of effective teaching of mathematics combines an understanding of how students learn, how to promote that learning by teaching through problem solving, and how to plan for and assess that learning on a daily basis. Introductory chapters in this section provide perspectives on trends in mathematics education and the process of doing mathematics. These chapters develop the core ideas of learning, teaching, planning, and assessment. Additional perspectives on mathematics for students with diverse backgrounds and the role of technological tools are also emphasized.



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## SECTION II Development of Mathematical Concepts and Procedures

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This section serves as the application of the core ideas of Section I. Here you will find chapters on every major content area in the pre-K–8 mathematics curriculum. Numerous problem-based activities to engage students are interwoven with a discussion of the mathematical content and how students develop their understanding of that content. At the outset of each chapter, you will find a listing of “Big Ideas,” the mathematical umbrella for the chapter. Also included are ideas for incorporating children’s literature, integrations with the mathematical practices, and formative assessment notes. These chapters are designed to help you develop pedagogical strategies and to serve as a resource for your teaching now and in the future.


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# Preface

All students can learn mathematics with understanding! It is through the teacher's actions that every student can have this experience. We believe that teachers must create a classroom environment in which students are given opportunities to solve problems and work together, using their ideas and strategies, to solve them. Effective mathematics instruction involves posing tasks that engage students in the mathematics they are expected to learn. Then, by allowing students to interact with and productively struggle with *their own mathematical ideas* and *their own strategies*, they will learn to see the connections among mathematical topics and the real world. Students value mathematics and feel empowered to use it.

Creating a classroom in which students design solution pathways, engage in productive struggle, and connect one mathematical idea to another is complex. Questions arise, such as, “How do I get students to wrestle with problems if they just want me to show them how to do it? What kinds of tasks lend themselves to this type of engagement? Where can I learn the mathematics content I need in order to be able to teach in this way?” With these and other questions firmly in mind, we have several objectives in the ninth edition of this textbook:

1. Illustrate what it means to teach mathematics using a problem-based approach.
2. Serve as a go-to reference for all of the mathematics content suggested for grades pre-K–8 as recommended in the Common Core State Standards (CCSSO, 2010) and in standards used in other states, and for the research-based strategies that illustrate how students best learn this content.
3. Present a practical resource of robust, problem-based activities and tasks that can engage students in the use of significant mathematical concepts and skills.
4. Report on technology that makes teaching mathematics in a problem-based approach more visible, including links to classroom videos and ready-to-use activity pages, and references to quality websites.

We hope you will find that this is a valuable resource for teaching and learning mathematics!

## NEW to this Edition

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We briefly describe new features below, along with the substantive changes that we have made since the eighth edition to reflect the changing landscape of mathematics education. The following are highlights of the most significant changes in the ninth edition.

### Blackline Masters, Activity Pages and Teacher Resource Pages

More than 130 ready-to-use pages have been created to support the problems and Activities throughout the book. By accessing the companion website, which lists the content by the page number in the text, you can download these to practice teaching an activity or to use with K–8 students in classroom settings. Some popular charts in the text have also been made into printable resources and handouts such as reflection questions to guide culturally relevant instruction.

### Activities at a Glance

By popular demand, we have prepared a matrix (Appendix D) that lists all Section II activities, the mathematics they develop, which CCSS standards they address, and the page where they can be found. We believe you will find this an invaluable resource for planning instruction.



## Self-Assessment Opportunities for the Reader

As we know, learners benefit from assessing their understanding along the way especially when there is a large amount of content to comprehend. To support teacher learning, each chapter begins with a set of learning outcomes that identify the goals of the chapter and link to Self-Check quizzes. Self-Checks fall at the end of every major text section. Also, at the end of each chapter the popular Writing to Learn section now has end-of-chapter questions.

## Expanded Lessons

Every chapter in Section II has at least one Expanded Lesson linked to an Activity. You may recognize some of these from the Field Experience Guide. These lessons focus on concepts central to elementary and middle school mathematics and include (1) NCTM and CCSSO grade-level recommendations, (2) adaptation suggestions for English language learners (ELLs) and students with special needs, and (3) formative assessment suggestions.

## Increased Focus on Common Core State Standards for Mathematics and Mathematical Practices

What began in the eighth edition is even stronger in the ninth edition. The CCSS are described in Chapter 1 along with other standards documents, and the Standards for Mathematical Practices are integrated into Chapter 2. In Section II, CCSS references are embedded in the text and every Activity lists the CCSS content that can be developed in that Activity. Standards for Mathematical Practice margin notes identify text content that shows what these practices look like in classroom teaching.

## Reorganization and Enhancement to Section I

If you are a seasoned user of this book, you will immediately note that Chapters 2 through 4 are dramatically different. Chapter 2 has Activity Pages for each of the tasks presented and the chapter has been reorganized to move theory to the end. Chapter 3 now focuses exclusively on worthwhile tasks and classroom discourse, with merged and enhanced discussion of problems and worthwhile tasks; the three-phase lesson plan format (*before, during, and after*) has been moved to the beginning of Chapter 4. Chapter 4, the planning chapter, also underwent additional, major revisions that include (1) adding in the lesson plan format, (2) offering a refined process for planning a lesson (now eight steps, not ten), and (3) stronger sections on differentiating instruction and involving families. Chapter 4 discussions about ELLs and students with special needs have been moved and integrated into Chapter 6. Chapter 7, on technology, no longer has content-specific topics but rather a stronger focus on emerging technologies. Content chapters now house technology sections as appropriate.

## Major Changes to Specific Chapters

---

### Basic Facts (Chapter 10)

There are three major changes to this chapter. First, there is a much stronger focus on assessing basic facts. This section presents the risks of using timed tests and presents a strong collection of alternative assessment ideas. Second, chapter discussions pose a stronger developmental focus. For example, the need to focus first on foundational facts before moving to derived facts is shared. Third, there is a shift from a focus on mastery to a focus on fluency (as described in CCSS and in the research).

### Developing Strategies for Addition and Subtraction (Chapters 11 and 12)

In previous editions there was a blurry line between Chapter 11 on place value and Chapter 12, which explored how to teach students to add and subtract. Although these topics overlap in many ways, we wanted to make it easier to find the appropriate content and corresponding activities. So, many components formerly in Chapter 11 (those that were explicitly about strategies for computing) have been shifted to Chapter 12 on addition and subtraction. This resulted in 15 more activities in Chapter 12, seven of which are new.

### Fraction Operations (Chapter 16)

Using learning trajectories and a developmental approach, the discussion of how to develop meaning for each operation has been expanded. For example, the operation situations presented in Chapter 9 are now connected in Chapter 16 to rational numbers. In particular, multiplication and division have received much more attention, including more examples and activities. These changes are in response to the many requests for more support in this area!

### Developing Concepts of Data Analysis (Chapter 21)

Look for several important changes in Chapter 21. There are 12 new activities that emphasize topics in CCSS. There also is more discussion on the shape of data, variability, and distribution. And, there is a notable increase in middle grades content including attention to dot plots, sampling, bivariate graphs, and, at the suggestion of reviewers, mean absolute deviation (MAD).

## Additional Important Chapter-Specific Changes

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The following substantive changes (not mentioned above) include

**Chapter 1:** Information about the new NCTM *Principles to Actions* publication with a focus on the eight guiding principles

**Chapter 2:** A revised and enhanced Doing Mathematics section and Knowing Mathematics section

**Chapter 3:** A new section on Adapting Tasks (to create worthwhile tasks) and new tasks and new authentic student work

**Chapter 4:** Open and parallel tasks added as ways to differentiate

**Chapter 5:** A more explicit development of how to use translation tasks to assess students' conceptual understanding

**Chapter 6:** Additional emphasis on multi-tiered systems of support including a variety of interventions

**Chapter 7:** Revisions reflect current software, tools, and digital apps as well as resources to support teacher reflection and collaboration

**Chapter 8:** Addition of Wright’s progression of children’s understanding of the number 10 and content from the findings from the new Background Research for the National Governor’s Association Center Project on Early Mathematics

**Chapter 9:** An expanded alignment with the problem types discussed in the CCSS document

**Chapter 13:** Expanded discussion of the written records of computing multiplication and division problems including lattice multiplication, open arrays, and partial quotients

**Chapter 14:** A reorganization to align with the three strands of algebraic thinking; a revamped section on Structure of the Number System with more examples of the connection between arithmetic and algebra; an increased focus on covariation and inequalities and a decreased emphasis on graphs and repeating patterns, consistent with the emphasis in CCSS

**Chapter 15:** Many fun activities added (with manipulatives such as Play-Doh, Legos, and elastic); expanded to increase emphasis on CCSS content, including emphasis on number lines and iteration

**Chapter 17:** Chart on common misconceptions including descriptions and examples

**Chapter 18:** Major changes to the Strategies section, adding tape diagrams and expanding the section on double number lines; increased attention to graphing ratios and proportions

**Chapter 19:** An increased focus on converting units in the same measurement system, perimeter, and misconceptions common to learning about area; added activities that explore volume and capacity

**Chapter 20:** The shift in organizational focus to the four major geometry topics from the precise van Hiele level (grouping by all level 1 components), now centered on moving students from level to level using a variety of experiences within a given geometry topic

**Chapter 22:** Major changes to activities and figures, an expanded focus on common misconceptions, and increased attention to the models emphasized in CCSS-M (dot plots, area representations, tree diagrams)

**Chapter 23:** A new section on developing symbol sense, expanded section on order of operations, and many new activities

## What You Will Find in This Book

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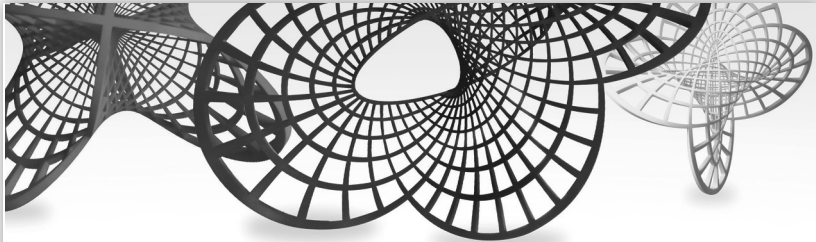
If you look at the table of contents, you will see that the chapters are separated into two distinct sections. The first section consists of seven chapters and covers important ideas that cross the boundaries of specific areas of content. The second section, consisting of 16 chapters, offers teaching suggestions and activities for every major mathematics topic in the pre-K–8 curriculum. Chapters in Section I offer perspectives on the challenging task of helping students learn mathematics. Having a feel for the discipline of mathematics—that is, to know what it means to “do mathematics”—is critical to learning how to teach mathematics well. In addition, understanding constructivist and sociocultural perspectives on learning mathematics and how they are applied to teaching through problem solving provides a foundation and rationale for how to teach and assess pre-K–8 students.

You will be teaching diverse students including students who are English language learners, are gifted, or have disabilities. In this text, you will learn how to apply instructional strategies in ways that support and challenge *all* learners. Formative assessment strategies, strategies for diverse learners, and effective use of technological tools are addressed in specific chapters in Section I (Chapters 5, 6, and 7, respectively), and throughout Section II chapters.

Each chapter of Section II focuses on one of the major content areas in pre-K–8 mathematics curriculum. It begins with identifying the big ideas for that content, and also provides guidance on how students best learn that content through many problem-based activities to engage them in understanding mathematics. Reflecting on the activities as you read can help you think about the mathematics from the perspective of the student. As often as possible, take out pencil and paper and try the problems so that you actively engage in *your learning* about *students learning* mathematics. In so doing, we are hopeful that this book will increase your own understanding of mathematics, the students you teach, and how to teach them well.

## Some Special Features of This Text

By flipping through the book, you will notice many section headings, a large number of figures, and various special features. All are designed to make the book more useful as a long-term resource. Here are a few things to look for.



**CHAPTER 15** **Developing Fraction Concepts**

**LEARNER OUTCOMES**

After reading this chapter and engaging in the embedded activities are able to:

- 15.1 Describe and give examples for fractions constructs.
- 15.2 Name the types of fractions models and describe activities
- 15.3 Explain foundational concepts of fractional parts, including  $\frac{1}{2}$  and connect these ideas to CCSS-M expectations.
- 15.4 Illustrate examples across fraction models for developing th
- 15.5 Compare fractions in a variety of ways and describe ways to
- 15.6 Synthesize how to effectively teach fraction concepts.

Fractions are one of the most important topics students need to understand in order to be successful in algebra and beyond, yet it is an area in which U.S. students struggle. NAEP test results have consistently shown that students have a weak understanding of fraction concepts (Sowder & Wearne, 2006; Wearne & Kouba, 2000). This lack of understanding is then translated into difficulties with fraction computation, decimal and percent concepts, and the use of fractions in other content areas, particularly algebra (Bailey, Hoard, Nugent, & Geary, 2012; Brown & Quinn, 2007; National Mathematics Advisory Panel, 2008). Therefore, it is absolutely critical that you teach fractions well, present fractions as interesting and important and commit to helping students understand the big ideas.

**BIG IDEAS**

- For students to really understand fractions, they must experience with various models and constructs, including part of a whole, ratios, and division.
- Three categories of models exist for working with fractions—area models, length (e.g.,  $\frac{3}{4}$  of an inch), and set or quantity (e.g.,  $\frac{3}{4}$  of the class).
- Partitioning and iterating are ways for students to understand the meaning of fractions, especially numerators and denominators.
- Equal sharing is a way to build on whole-number knowledge to understand fractions.
- Equivalent fractions are ways of describing the same amount by different names.
- Fractions can be compared by reasoning about the relative size of the parts and reasoning are important in teaching understanding of fractions.

◀ **Learning Outcomes [NEW]**

To help readers know what they should expect to learn, each chapter begins with learning outcomes. Self-checks are numbered to cover and thus align with each learning outcome.

◀ **Big Ideas**

Much of the research and literature espousing a student-centered approach suggests that teachers plan their instruction around big ideas rather than isolated skills or concepts. At the beginning of each chapter in Section II, you will find a list of the key mathematical ideas associated with the chapter. Teachers find these lists helpful to quickly envision the mathematics they are to teach.

**Self-Check Prompts [NEW] ▶**

To help readers self-assess what they have just read, a self-check prompt is offered at the end of each significant text section. After answering these quiz questions online and submitting their responses, users can review feedback on what the correct response is (and why).

You may decide instead to break the shape up into two rectangles and ask the student to find the area of each shape and combine. Then have the student attempt the next shape without the modification—you should always lead back to the original task. However, if you decide to begin with rectangular regions and build to compound shapes composed of rectangles, you have *self-folded* the lesson in a way to ramp up to the original task. In planning accommodations and modifications, the goal is to enable each student to successfully reach your learning objectives, not to change the objectives. This is how equity is achieved—by reaching equal *outcomes*, not by equal treatment. Treating students the same when they each learn differently does not make sense.

Complete an Accommodation or Modification Needs table to reflect on how you will plan for students in your classroom who have special needs. Record the evidence that you are adapting the learning situation.

Complete Self-Check 6.1: Mathematics for ALL Students

**Providing for Students Who Struggle and Those with Special Needs**

One of the basic tenets of education is the need for individualizing the content taught and the methods used for students who struggle, particularly those with special needs. Mathematics learning disabilities are best thought of as cognitive differences, not cognitive deficits (Lewis, 2014). Students with disabilities often have mandated individualized education programs (IEPs) that guarantee access to grade-level mathematics content—in a general education classroom, if possible. This legislation also implies that educators consider individual learning needs not only in terms of *what* mathematics is taught but also *how* it is taught.

**Prevention Models**

In many areas, a systematic process for achieving higher levels performance for all students includes a multitiered system of support frequently called response to intervention (RtI). This

**Activity 17.2**

CCSS-M: 4.NF.C.6; 5.NBT.A.1; 5.NBT.A.2; 5.NBT.A.3a

**Shifting Units**

Give students a collection of paper base-ten pieces created from Base-Ten Materials, or base-ten blocks. Ask them to pull out a particular mix—for example, a student might have three squares, seven strips, and four “tinies.” Tell students that you have the unit behind your back; when you show it to them, they are to figure out how much they have and to record the value. Hold up one of the units. Observe what students record as their value. Ask students to accurately say their quantity aloud. For ELLs and students with disabilities, it is particularly important that you write these labels with the visuals in a prominent place in the classroom (and in student notebooks) so that they can refer to the terminology and illustrations as they participate in the activity. Repeat several times. Be sure to include examples in which a piece is not represented so that students will understand decimal values like 0.07. Continue played in partners with one student selecting a mix of the number.



**▲ Activities**

The numerous activities found in every chapter of Section II have always been rated by readers as one of the most valuable parts of the book. Some activity ideas are described directly in the text and in the illustrations. Others are presented in the numbered Activity boxes. Every activity is a problem-based task (as described in Chapter 3) and is designed to engage students in doing mathematics.

Numbers like 3.2 or 12.1389 do not relate to money and can cause confusion (Martinez, 2007). Students’ initial contact with decimals should be more flexible, and so money is not recommended as an initial model for decimals, although it is certainly an important application of

**◀ Adaptations for Students with Disabilities and English Language Learners**

Chapter 6 provides detailed background and strategies for how to support students with disabilities and English language learners (ELLs). But, many adaptations are specific to a particular activity or task. Therefore, Section II chapters offer activities (look for the icon) that can meet the needs of exceptional students including specific instructions with adaptations directly within the Activities.

**Formative Assessment Notes ▶**

Assessment should be an integral part of instruction. Similarly, it makes sense to think about what to be listening for (assessing) as you read about different areas of content development. Throughout the content chapters, there are formative assessment notes with brief descriptions of ways to assess the topic in that section. Reading these assessment notes as you read the text can also help you understand how best to assist struggling students.

**FORMATIVE ASSESSMENT Notes.** When are students ready to work on reasoning strategies? When they are able to (1) use counting-on strategies (start with the largest and count up) and (2) see that numbers can be decomposed (e.g., that 6 is 5 + 1). Interview students by posing one-digit addition problems and ask how they solved it. For example, 3 + 8 (Do they count on from the larger?) and 5 + 6 (Do they see 5 + 5 + 1?). For multiplication, 3 × 8 (Do they know this is 3 eights? Do they see it as 2 eights and one more eight?). ■

Complete Self-Check 10.1: Developmental Phases for Learning the Basic Facts



**Teaching and Assessing the Basic Facts**

This section describes the different ways basic fact instruction has been implemented in schools, followed by a section describing effective strategies.

**Different Approaches to Teaching the Basic Facts**

Over the last century, three main approaches have been used to teach the basic facts. The pros and cons of each approach are briefly discussed in this section.

**Memorization.** This approach moves from presenting concepts of addition and multiplication straight to memorization of facts, not devoting time to developing strategies (Baroody, Baiwa, & Eiland, 2009). This approach requires students to memorize 100 separate addition facts (just for the addition combinations 0–9) and 100 multiplication facts (0–9). Students may even have to memorize subtraction and division separately—bringing the total to over 300

### Creating Graphs

Students should be involved in deciding how they want to represent their data, but they will need to be introduced to what the options are and when each display can and cannot be used.

The value of having students actually construct their own graphs is not so much that they learn the techniques, but that they are personally invested in the data and that they learn how a graph conveys information. Once a graph is constructed, the most important activity is discussing what it tells the people who see it. Analyzing data that are numerical (number of pockets) versus categorical (color of socks) is an added challenge for students as they struggle to make sense of the graphs (Russell, 2006). If, for example, the graph has seven stickers above the five, students may think that five people have seven pockets or seven people have five pockets.

Creating graphs requires care and precision, including determining appropriate scales and labels. But the reason for the precision is so that an audience is able to see at a glance the summary of the data gathered on a particular question.

**TECHNOLOGY Note.** Computer programs and graphing calculators can provide a variety of graphical displays. Use the time saved by technology to focus on the discussions about the information that each display provides! Students can make their own selections from among different graphs and justify their choice based on their own intended purposes. The graphing calculator puts data analysis technology in the hands of every student. The TI-73 calculator is designed for middle-grade students. It will produce eight different kinds of plots or graphs, including pie charts, bar graphs, and picture graphs, and will compute and graph lines of best fit. The Internet also offers opportunities to explore different graphs. Create a Graph (NCES Kids

**CCSS** Standards for Mathematical Practice  
**MP6.** Attend to precision.

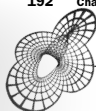
**CCSS** Standards for Mathematical Practice  
**MP5.** Use appropriate tools strategically.

### Standards for Mathematical Practice Margin Notes [NEW]

Connections to the eight Standards of Mathematical Practice from the Common Core State Standards are highlighted in the margins. The location of the note indicates an example of the identified practice in the nearby text.

### Technology Notes

Infusing technological tools is important in learning mathematics, as you will learn in Chapter 7. We have infused technology notes throughout Section II. A technology icon is used to identify places within the text or activity where a technology idea or resource is discussed. Descriptions include open-source (free) software, applets, and other Web-based resources, as well as ideas for calculator use.



## REFLECTIONS ON CHAPTER 9

### WRITING TO LEARN

Click here to assess your understanding and application of chapter content.

1. Make up a compare story problem. Alter the problem to provide examples of all six different possibilities compare problems.
2. Explain how missing-part activities prepare students mastering subtraction facts.
3. Make up multiplication story problems to illustrate difference between equal groups and multiplicative comparison. Then create a story problem involving rate area or arrays.
4. Why is the use of key words not a good strategy to tell children?

### FOR DISCUSSION AND EXPLORATION

### End of Chapter Resources

The end of each chapter includes two major subsections: *Reflections*, which includes “Writing to Learn” and “For Discussion and Exploration,” and *Resources*, which includes “Literature Connections” (found in all Section II chapters) and “Recommended Readings.”

**Writing to Learn [ENHANCED].** Questions are provided that help you reflect on the important pedagogical ideas related to the content in the chapter. Actually writing out the answers to these questions in your own words, or talking about them with peers, is one of the best ways for you to develop your understanding of each chapter’s main ideas.

**For Discussion and Exploration.** These questions ask you to explore an issue related to that chapter’s content, applying what you have learned. For example, questions may ask you to reflect on classroom observations, analyze curriculum materials, or take a position on controversial issues. We hope that these questions will stimulate thought and cause spirited conversations.

**Literature Connections.** Section II chapters contain great children’s literature for launching into the mathematics concepts in the chapter just read. For each title suggested, there is a brief description of how the mathematics concepts in the chapter can be connected to the story. These literature-based mathematics activities will help you engage students in interesting contexts for doing mathematics.

**Recommended Readings.** In this section, you will find an annotated list of articles and books to augment the information found in the chapter. These recommendations include NCTM articles and books, and other professional resources designed for the classroom teacher. (In addition to the Recommended Readings, there is a References list at the end of the book for all sources cited within the chapters.)



## RESOURCES FOR

### LITERATURE CONNECTIONS

There are many books with stories or pictures concern collections, the purchase of items, measurements, and so that can be used to pose problems or, better, to stimulate children to invent their own problems. Perhaps the most mentioned book in this context is *The Doorbell Rang* by Hutchins (1986). You can check that one out yourself, as well as the following suggestions.

#### Bedtime Math Overdeck (2013)

This book (and accompanying website) is the author’s attempt to get parents to incorporate math problems into nighttime (or daytime) routine. There are three levels of difficulty, starting with problems for “wee ones” (pre-K), “big kids” (K–2), and “big kids” (grade 2 and up). Each set of problems revolves around a high-interest topic such as roller coasters, foods, and animals. Teachers can use these problems in class for engaging students in all four operations.

#### One Hundred Hungry Ants Pinczes (1999) View One Hundred Angry Ants (https://www.youtube.com/watch?v=kmdSUHPwJtc)

#### A Remainder of One Pinczes (2002)

These two books, written by a grandmother for her grandchild, help students explore multiplication and division.

## Supplements for Instructors

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Qualified college adopters can contact their Pearson sales representatives for information on ordering any of the supplements described below. These instructor supplements are all posted and available for download (click on Educators) from the Pearson Instructor Resource Center at [www.pearsonglobaleditions.com/vandewalle](http://www.pearsonglobaleditions.com/vandewalle). The IRC houses the following:

- **Instructor’s Resource Manual** The Instructor’s Resource Manual for the ninth edition includes a wealth of resources designed to help instructors teach the course, including chapter notes, activity suggestions, and suggested assessment and test questions.
- **Electronic Test Bank** An electronic test bank (TB) contains hundreds of challenging questions as multiple-choice or short-answer questions. Instructors can choose from these questions and create their own customized exams.
- **PowerPoint™ Presentation** Ideal for instructors to use for lecture presentations or student handouts, the PowerPoint presentation provides ready-to-use graphics and text images tied to the individual chapters and content development of the text.

## Acknowledgments

---

Many talented people have contributed to the success of this book, and we are deeply grateful to all those who have assisted over the years. Without the success of the first edition, there would certainly not have been a second, much less nine editions. The following people worked closely with John on the first edition, and he was sincerely indebted to Warren Crown, John Dossey, Bob Gilbert, and Steven Willoughby, who gave time and great care in offering detailed comments on the original manuscript.

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*From Karen Karp:* I would like to express thanks to my husband, Bob Ronau, who as a mathematics educator graciously helped me think about decisions while offering insights and encouragement. In addition, I thank my children, Matthew, Tammy, Joshua, Misty, Matt, Christine, Jeffrey, and Pamela for their kind support and inspiration. I also am grateful for my wonderful grandchildren, Jessica, Zane, Madeline, Jack and Emma, who have helped deepen my understanding about how children think.

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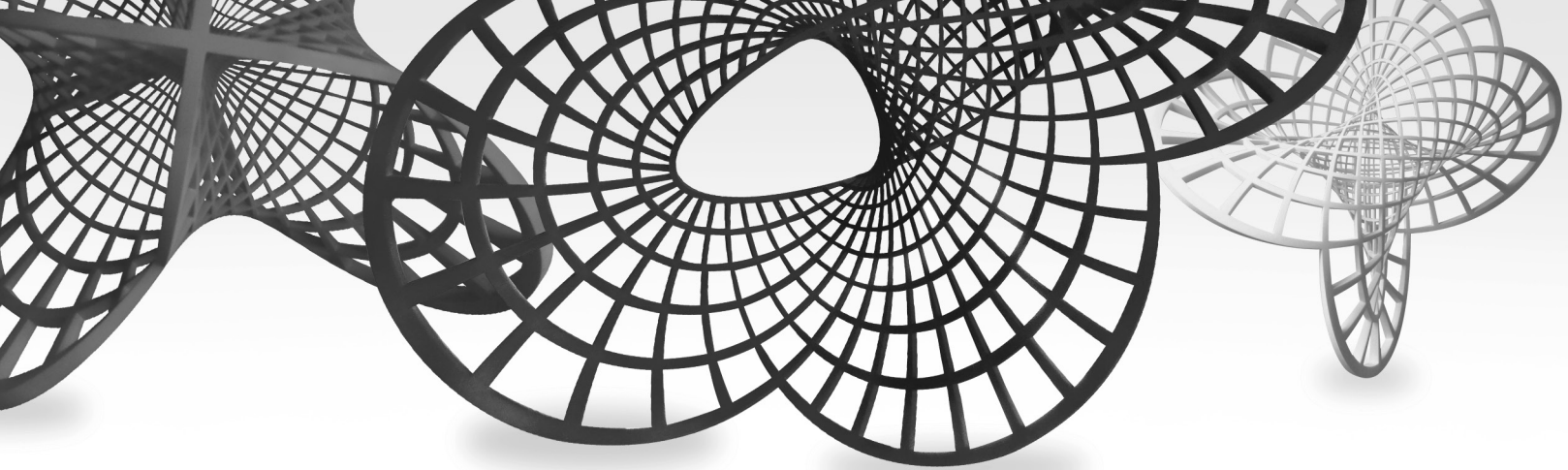
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## CHAPTER

# 1

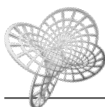
# Teaching Mathematics in the 21st Century

## LEARNER OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- 1.1 Summarize the factors that influence the teaching of mathematics.
- 1.2 Describe the important documents that are a part of the movement toward a set of shared expectations for students.
- 1.3 Explore the qualities needed to learn and grow as a professional teacher of mathematics.

Someday soon you will find yourself in front of a class of students, or perhaps you are already teaching. What general ideas will guide the way you will teach mathematics? This book will help you become comfortable with the mathematics content of the pre-K–8 curriculum. You will also learn about research-based strategies for helping students come to know mathematics and be confident in their ability to do mathematics. These two things—your knowledge of mathematics and how students learn mathematics—are the most important tools you can acquire to be successful.



## Becoming an Effective Teacher of Mathematics

Before we get started, think back to when you were in pre-K–8 classrooms as a student. What are your remembrances of learning mathematics? Here are some thoughts from in-service and pre-service teachers of whom we asked the same question. Which description do you resonate with?

I was really good at math in lower elementary grades, but because I never understood why math works, it made it very difficult to embrace the concepts as I moved into higher grades. I started believing I wasn't good at math so I didn't get too upset when my grades reflected that. *Kathryn*

As a student I always felt lost during mathematics instruction. It was as if everyone around me had a magic key or code that I missed out on getting. *Tracy*

I remember math being very challenging, intimidating, and capable of making me literally sick to my stomach. Math was a bunch of rules and formulas I was expected to memorize, but not to understand. *Mary Rebekah*

I consider myself to be really good at math and I enjoy mathematics-related activities, but I often wonder if I would have been GREAT at math and had a completely different career if I cared about math as much as I do now. Sometimes I feel robbed. *April*

Math went from engaging, interactive instruction that I excelled at and loved, to lecture-style instruction that I struggled with. I could not seek outside help, even though I tried, because the teacher's way was so different from the way of the people trying to help me. I went from getting all As to getting low Bs and Cs without knowing how the change happened.

*Janelle*

Math class was full of elimination games where students were pitted against each other to see who could answer a math fact the fastest. Because I have a good memory I did well, but I hated every moment. It was such a nerve-wracking experience and for the longest time that is what I thought math was. *Lawrence*

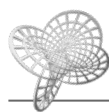
Math was never a problem because it was logical, everything made sense. *Tova*

As you can see these memories run the gamut with an array of emotions and experiences. The question now becomes, what do you hope your students will say as they think back to your mathematics instruction? The challenge is to get all of your students to learn mathematics with understanding and enthusiasm. Would you relish hearing your students, fifteen years after leaving your classroom, state that you encouraged them to be mathematically minded, curious about solving new problems, self-motivated, able to critically think about both correct and incorrect strategies, and that you nurtured them to be a risk takers willing to try and persevere on challenging tasks? What will your legacy be?

As part of your personal desire to build successful learners of mathematics, you might recognize the challenge that mathematics is sometimes seen as the subject that people love to hate. At social events of all kinds—even at parent-teacher conferences—other adults will respond to the fact that you are a teacher of mathematics with comments such as “I could never do math,” or “I can't even balance my checking account.” Instead of dismissing these disclosures, consider what positive action you can take. Would people confide that they don't read and hadn't read a book in years? That is not likely. Families' and teachers' attitudes toward mathematics may enhance or detract from students' ability to do math. It is important for you and for students' families to know that mathematics ability is not inherited—anyone can learn mathematics. Moreover, learning mathematics is an essential life skill. You need to find ways of countering these statements, especially if they are stated in the presence of students, pointing out that it is a myth that only some people can be successful in learning mathematics. Only in that way can the chain of passing apprehension from family member to child, or in rare cases teacher to student, be broken. There is much joy to be had in solving mathematical problems, and you need to model this excitement and nurture that passion in your students.

Your students need to ultimately think of themselves as mathematicians in the same way as many of them think of themselves as readers. As students interact with our increasingly mathematical and technological world, they need to construct, modify, communicate or integrate new information in many forms. Solving novel problems and approaching circumstances with a mathematical perspective should come as naturally as reading new materials to comprehend facts, insights, or news. Consider how important this is to interpreting and successfully surviving in our economy and in our environment.

The goal of this book is to help you understand the mathematics methods that will make you an effective teacher. As you dig into the information your vision and confidence will grow.



## A Changing World

In his book *The World Is Flat* (2007), Thomas Friedman discusses the need for people to have skills that are lasting and will survive the ever-changing landscape of available jobs. These are specific categories within a larger group that are called “untouchables” as regardless of the shifting landscape of job options—they will be successful in finding jobs. He is the one who defined these broad categories—such as math lover. Friedman points out that in a world that is digitized

and surrounded by algorithms, math lovers will always have career opportunities and options. This is important as science, technology, engineering, and math (STEM) jobs, because of a skills gap, take more than twice as long to fill as other jobs in the marketplace (Rothwell, 2014). This is also aligned with the thinkers who believe students need to not just be college ready but innovation ready (Wagner, 2012).

Now it becomes the job of every teacher of mathematics to prepare students with skills for potential careers and develop a “love of math” in students. Lynn Arthur Steen, a well-known mathematician and educator, stated, “As information becomes ever more quantitative and as society relies increasingly on computers and the data they produce, an innumerate citizen today is as vulnerable as the illiterate peasant of Gutenberg’s time” (1997, p. xv).

The changing world influences what should be taught in pre-K–8 mathematics classrooms. As we prepare pre-K–8 students for jobs that possibly do not currently exist, we can predict that there are few jobs for people where they just do simple computation. We can also predict that there will be work that requires interpreting complex data, designing algorithms to make predictions, and using the ability to approach new problems in a variety of ways.

As you prepare to help students learn mathematics for the future, it is important to have some perspective on the forces that effect change in the mathematics classroom. This chapter addresses the leadership that you, the teacher, will develop as you shape the mathematics experience for your students. Your beliefs about what it means to know and do mathematics and about how students make sense of mathematics will affect how you approach instruction and the understandings and skills your students take from the classroom.

## Factors to Consider

For more than two decades, mathematics education has constantly undergone change. There have been significant reforms that reflect the technological and informational needs of our society, research on how students learn mathematics, the importance of providing opportunities to learn for all students, and ideas on how and what to teach from an international perspective. Just as we would not expect doctors to be using the exact same techniques and medicines that were prevalent when you were a child, teachers’ methods are evolving and transforming via a powerful collection of expert knowledge about how the mind functions and how to design effective instruction (Wiggins, 2013).

There are several significant factors in this transformation. One factor is the public or political pressure for change in mathematics education due largely to information about student performance in national and international studies. These large scale comparisons of student performance continue to make headlines, provoke public opinion, and pressure legislatures to call for tougher standards backed by testing. The pressures of testing policies exerted on schools and ultimately on teachers may have an impact on instruction. These studies are important because international and national assessments provide strong evidence that mathematics teaching *must* change if our students are to be competitive in the global market and able to understand the complex issues they must confront as responsible citizens.

**National Assessment of Education Progress (NAEP).** Since the 1960s, at regular intervals, the United States gathers data on how fourth-, eighth-, and twelfth-grade students are doing in mathematics on the NAEP. These data provide an important tool for policy makers and educators to measure the overall improvement of U.S. students over time in what is called the “Nation’s Report Card.” NAEP uses four achievement levels: below basic, basic, proficient, and advanced, with proficient and advanced representing substantial grade-level achievement. The criterion-referenced test is designed to reflect the current curriculum but keeps a few stable items from 1982 for purposes of comparison (Kloosterman, Rutledge, & Kenney, 2009b). In the most recent assessment in 2013, less than half of all U.S. students in grades 4 and 8 performed at the desirable levels of proficient and advanced (42 percent in fourth grade and 35 percent in eighth grade) (National Center for Education Statistics, 2013). Despite encouraging gains in the NAEP scores over the last 30 years due to important shifts in instructional practices (particularly at the elementary level) (Kloosterman, Rutledge, & Kenney, 2009b), some U.S. students’ performance still reveals disappointing levels of competency.

**Trends in International Mathematics and Science Study (TIMSS).** In the mid-1990s, 41 nations participated in the Third International Mathematics and Science Study, the largest study of mathematics and science education ever conducted. Data were gathered in grades 4, 8, and 12 from 500,000 students as well as from teachers. The most widely reported results revealed that U.S. students performed above the international average of the TIMSS countries at the fourth grade, below the average at the eighth grade, and significantly below average at the twelfth grade (National Academy Press, 1999; U.S. Department of Education, 1997).

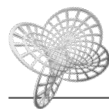
TIMSS studies were repeated often with the most recent in 2011 in which 57 countries participated. For details, please visit the TIMSS website. The 2011 TIMSS found that U.S. fourth and eighth graders were above the international average but were significantly outperformed at fourth-grade level mathematics by education systems in Singapore, Republic of Korea, Hong Kong, Chinese Taipei, Japan, Northern Ireland, Belgium, Finland, England, and the Russian Federation and outperformed at the eighth-grade level by education systems in Republic of Korea, Singapore, Chinese Taipei, Hong Kong, Japan, Russian Federation, Israel, and Finland.

One of the most interesting components of the study was the videotaping of eighth-grade classrooms in the United States, Australia, and five of the highest-achieving countries. The results indicate that teaching is a cultural activity and, despite similarities, the differences in the ways countries taught mathematics were often striking. In all countries, problems or tasks were frequently used to begin the lesson. However, as a lesson progressed, the way these problems were handled in the United States was in stark contrast to high-achieving countries. Analysis revealed that, although the world is for all purposes unrecognizable from what it was 100 years ago, the U.S. approach to teaching mathematics during the same time frame was essentially unchanged (Stigler & Hiebert, 2009). Other countries incorporated a variety of methods, but they frequently used a problem-solving approach with an emphasis on conceptual understanding and students engaged in problem solving (Hiebert et al., 2003). Teaching in the high-achieving countries more closely resembles the recommendations of the National Council of Teachers of Mathematics, the major professional organization for mathematics teachers, discussed next.

**National Council of Teachers of Mathematics (NCTM).** One transformative factor is the professional leadership of the National Council of Teachers of Mathematics (NCTM). The NCTM, with more than 80,000 members, is the world's largest mathematics education organization. This group holds an influential role in the support of teachers and an emphasis on what is best for learners. Their guidance in the creation and dissemination of standards for curriculum, assessment, and teaching led the way for other disciplines. For an array of resources, including the Illuminations component which consists of a set of exciting instructional experiences for your students, visit the NCTM website.



### Complete Self-Check 1.1: A Changing World



## The Movement toward Shared Standards

The momentum for reform in mathematics education began in earnest in the early 1980s. The main impetus was a response to a need for more problem solving as well as the research of developmental psychologists who identified how students can best learn mathematics. Then in 1989, NCTM published the first set of standards for a subject area in the *Curriculum and Evaluation Standards for School Mathematics*. Many believe that no other document has ever had such an enormous effect on school mathematics or on any other area of the curriculum.

NCTM followed in 1991 with a set of standards for teaching that articulated a vision of teaching mathematics for all students, not just a few. In 1995, NCTM added to the collection the *Assessment Standards for School Mathematics*, which focused on the importance of integrating

assessment with instruction and indicated the key role that assessment plays in implementing change (see Chapter 5). In 2000, however, NCTM released *Principles and Standards for School Mathematics* as an update of its original standards document. Combined, these documents prompted a revolutionary reform movement in mathematics education throughout the world.

As these documents influenced teacher practice, ongoing debate continued about the U.S. curriculum. In particular, many argued that instead of hurrying through several topics every year, the curriculum needed to address content more deeply. Guidance was needed in deciding what mathematics content should be taught at each grade level and, in 2006, NCTM released *Curriculum Focal Points*, a little publication with a big message—the mathematics taught at each grade level needs to be focused, provide depth, and explicitly show connections.

In 2010, the Council of Chief State School Officers (CCSSO) presented the Common Core State Standards, which are grade-level specific standards which incorporate ideas from *Curriculum Focal Points* as well as international curriculum documents. A large majority of U.S. states adopted these as their standards. In less than 25 years, the standards movement transformed the country from having little to no coherent vision on what mathematics should be taught and when, to a more widely shared idea of what students should know and be able to do at each grade level.

In the following sections, we discuss three significant documents critical to your work as a teacher of mathematics.

## Principles and Standards for School Mathematics

The *Principles and Standards for School Mathematics* (NCTM, 2000) provides guidance and direction for teachers and other leaders in pre-K–12 mathematics education. This is particularly true in states and regions where they have developed their own standards.

**The Six Principles.** One of the most important features of *Principles and Standards for School Mathematics* is the articulation of six principles fundamental to high-quality mathematics education. These principles must be blended into all programs as building excellence in mathematics education involves much more than simply listing content objectives.

**The Equity Principle.** The strong message of this principle is there should be high expectations and intentional ways to support all students. All students must have the opportunity and adequate support to learn mathematics regardless of their race, socioeconomic status, gender, culture, language, or disability. This principle is interwoven into all other principles.

**The Curriculum Principle.** The curriculum should be coherent and built around “big ideas” in the curriculum and in daily classroom instruction. We think of these big ideas as “important” if they help develop other ideas, link one idea to another, or serve to illustrate the discipline of mathematics as a human endeavor. Students must be helped to see that mathematics is an integrated whole that grows and connects across the grades rather than a collection of isolated bits and pieces.

**The Teaching Principle.** What students learn about mathematics depends almost entirely on the experiences that teachers provide every day in the classroom. To provide high-quality mathematics education, teachers must (1) understand deeply the mathematics content they are teaching; (2) understand how students learn mathematics, including common misconceptions; and (3) select meaningful instructional tasks and generalizable strategies that will enhance learning.

**The Learning Principle.** This principle is based on two fundamental ideas. First, learning mathematics with understanding is essential. Mathematics today requires not only computational skills but also the ability to think and reason mathematically to solve new problems and learn to respond to novel situations that students will face in the future. Second, students *can* learn mathematics with understanding. Learning is enhanced in classrooms where students are required to evaluate their own ideas and those of others, make mathematical conjectures and test them, and develop their reasoning and sense-making skills.

**The Assessment Principle.** Ongoing assessment highlights the most important mathematics concepts for students. Assessment that includes ongoing observation and student interaction

encourages students to articulate and, thus, clarify their ideas. Feedback from daily assessment helps students establish goals and become more independent learners. By continuously gathering data about students' understanding of concepts and growth in reasoning, teachers can better make the daily decisions that support student learning. For assessment to be effective, teachers must use a variety of assessment techniques, understand their mathematical goals deeply, and have a research-supported notion of students' thinking or common misunderstandings.

**The Technology Principle.** Calculators, computers, and other emerging technologies are essential tools for learning and doing mathematics. Technology permits students to focus on mathematical ideas, to reason, and to solve problems in ways that are often impossible without these tools. Technology enhances the learning of mathematics by allowing for increased exploration, enhanced representation, and communication of ideas.

**The Five Content Standards.** *Principles and Standards* includes four grade bands: pre-K–2, 3–5, 6–8, and 9–12. The emphasis on preschool recognizes the need to highlight the critical years before students enter kindergarten. There is a common set of five content standards throughout the grades:

- Number and Operations
- Algebra
- Geometry
- Measurement
- Data Analysis and Probability

Each content standard includes a set of goals applicable to all grade bands followed by specific expectations for what students should know at each grade band. Although the same five content standards apply across all grades, you should not infer that each strand has equal weight or emphasis in every grade band. Number and Operations is the most heavily emphasized strand from pre-K through grade 5 and continues to be important in the middle grades, with a lesser emphasis in grades 9–12. This is in contrast to Algebra, which moves from an emphasis related to number and operations in the early grades and builds to a strong focus in the middle and high school grade bands.

**The Five Process Standards.** The process standards refer to the mathematical processes through which pre-K–12 students acquire and use mathematical knowledge. The process standards should not be regarded as separate content or strands in the mathematics curriculum, rather, they are integral components of all mathematics learning and teaching. The five process standards and ways you can develop these elements in your students can be found in Table 1.1.

Members of NCTM have free online access to the *Principles and Standards* and nonmembers can sign up for 120 days of free access to the full document on the NCTM website under the tab Standards and Focal Points.

## Common Core State Standards

As noted earlier, the dialogue on improving mathematics teaching and learning extends beyond mathematics educators. Policymakers and elected officials considered previous NCTM standards documents, international assessments, and research on the best way to prepare students to be “college and career ready.” The National Governors Association Center for Best Practices and the Council of Chief State School Officers (CCSSO) collaborated with other professional groups and entities to develop shared expectations for K–12 students across states, a focused set of mathematics content standards and practices, and efficiency of material and assessment development (Porter, McMaken, Hwang, & Yang, 2011). As a result, they created the Common Core State Standards for Mathematics (CCSS-M) which can be downloaded for free at <http://www.corestandards.org/math>. At this time more than 40 states, Washington, D.C., four territories, and Department of Defense Schools have adopted the Common Core State Standards. This represents the largest shift of mathematics content in the United States in more than 100 years. A few states did not opt to participate in the adoption of the standards from the start of their development and at this time others are still deciding their level of participation or reevaluating their own standards against CCSS-M.